

Tau Function of the CKP Hierarchy and Non-linearizable Virasoro Symmetries

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Abstract

We introduce a single tau function that represents the CKP hierarchy into a generalized Hirota “bilinear” equation. The actions on the tau function by additional symmetries for the hierarchy are calculated, which involve strictly more than a central extension of the w_∞^C -algebra. As an application, for Drinfeld-Sokolov hierarchies of type C that are equivalent to certain reductions of the CKP hierarchy, their Virasoro symmetries are proved to be non-linearizable when acting on the tau function.

Key words: tau function; CKP hierarchy; Drinfeld–Sokolov hierarchy; Virasoro symmetry

1 Introduction

The Kadomtsev-Petviashvili (KP) hierarchy together with its subhierarchies of types B and C [3, 2], abbreviated as the BKP and the CKP hierarchies respectively, has attracted much research interest in areas of mathematical physics. These hierarchies can be represented equivalently as Lax equations of pseudo-differential operators or as bilinear equations. For instance, the CKP hierarchy concerned in the present paper is defined by the following bilinear equation

$$\text{res}_z w(\mathbf{t}; z) w(\mathbf{t}'; -z) = 0, \quad (1.1)$$

where w is the so-called wave function depending on the time variable $\mathbf{t} = (t_1, t_3, t_5, \dots)$ and a parameter z , and $\text{res}_z \sum_i f_i z^i = f_{-1}$ for any formal Laurent series in z .

For the CKP hierarchy, in contrast to the KP and the BKP cases, there seems not a single tau function that represents (1.1) to the form of Hirota bilinear equations, though it was pointed out by Date, Jimbo, Kashiwara and Miwa [2] that a tau function may be constructed from the action of bosonic fields on the vacuum vector in a Fock space. The idea in [2] was developed by van de Leur, Orlov and Shiota [13] later.

They introduced a series of fermi operators besides the bosonic fields, and constructed a tau function depending on both time variables t_k and certain odd Grassmannian parameters. In fact it is a tau function of a generalization of the CKP hierarchy, i.e., a system of bilinear equations like (1.1) of wave functions labeled with “odd number of Odd Partitions with Distinct parts” (see equation (2.57) in [13]). In particular, their wave function w_1 (with its expansion (2.53)–(2.55) in [13]) solves the bilinear equation (1.1) of the CKP hierarchy, but it is related to a series of rather than only one tau function.

In this paper we are to introduce a tau function $\tau(\mathbf{t})$ of the CKP hierarchy by making use of its Hamiltonian densities, in consideration of that the hierarchy carries a series of bi-Hamiltonian structures reduced from those for the KP hierarchy [5]. This tau function will be shown related to the wave function via the following formula

$$w(\mathbf{t}; z) = \left(1 + \frac{1}{z} \frac{\partial}{\partial t_1} \log \frac{G(z)\tau(\mathbf{t})}{\tau(\mathbf{t})} \right)^{1/2} \frac{G(z)\tau(\mathbf{t})}{\tau(\mathbf{t})} e^{\xi(\mathbf{t}; z)}, \quad (1.2)$$

where

$$G(z) = \exp \left(- \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{2}{k} \frac{\partial}{\partial t_k} \right), \quad \xi(\mathbf{t}; z) = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} t_k z^k.$$

Observe that (1.2) is different from those formulae relating wave and tau functions in the literature, namely, now there is a square-root factor depending on the tau function. By substituting this formula into (1.1), the CKP hierarchy is recast to a generalized Hirota “bilinear” equation of tau function (see equation (3.19) below).

For the CKP hierarchy, its additional symmetries were constructed by He, Tian Foerster and Ma [8] with the help of Orlov-Schulman operators [15]. They also showed that these additional symmetries acting on the wave function $w(\mathbf{t}; z)$ form a centerless w_∞^C -algebra. As observed by Adler, Shiota and van Moerbeke, the lifting of the actions of additional symmetries on wave function to that on tau function results in a central extension of the sort of w_∞ -algebra. Such phenomenon were confirmed for the KP and the BKP hierarchies as well as for the two-dimensional Toda lattice and the two-component BKP hierarchies [1, 4, 12, 16, 18]. However, a counter example will be found, that is the CKP hierarchy (1.1). For this hierarchy, when the additional symmetries act on the tau function in (1.2), it implies not only a central extension of the w_∞^C -algebra but also some non-trivial “tails” given by polynomials in at-least-second-order derivatives of $\log \tau$ with respect to the time variables. So far as we know, such kind of examples have not been considered in the literature before.

This paper is also motivated by the study of non-linearizable Virasoro symmetries for integrable hierarchies proposed by one of the authors in [19] when considering Drinfeld-Sokolov hierarchies. Recall that for every affine Kac-Moody algebra with an arbitrary vertex of the Dynkin diagram marked (only the case of the zeroth vertex is considered below), Drinfeld and Sokolov [6] constructed an integrable hierarchy of Korteweg-de Vries (KdV) type. These hierarchies are (or expected to be) applied to various areas of mathematical physics, while studying their Virasoro symmetries provides an important approach to such applications. For instance, one of the main

properties of integrable hierarchies of topological type [7] is that their Virasoro symmetries act linearly on the tau function. This property is valid [19] (see also [17, 7, 18]) for the Drinfeld-Sokolov hierarchy associated to each ADE-type affine Kac-Moody algebra, either untwisted or twisted [10], which completes the proof of a conjecture raised by Dubrovin and Zhang [7]. In contrast, Drinfeld-Sokolov hierarchies associated to affine Kac-Moody algebras of the other types were claimed [19] not to have linearized Virasoro symmetries. We will prove this argument for the hierarchies of type $C_n^{(1)}$ with $n \geq 2$. Our method is to consider that each of these hierarchies is equivalent to a certain reduction of the CKP hierarchy and that its Virasoro symmetries are reduced accordingly from the additional symmetries for the latter.

To achieve these results, we first recall the definition of the CKP hierarchy and its additional symmetries in the following section. In Section 3 we introduce a tau function of the CKP hierarchy by using its Hamiltonian densities and then represent the hierarchy into a bilinear equation of tau function. The actions on tau function by the additional symmetries is considered in Section 4. Section 5 consists of two parts. The first part is devoted to a brief review of the construction of the Drinfeld-Sokolov hierarchies from affine Kac-Moody algebras of type C as well as their tau function and Virasoro symmetries; in the second part, these Virasoro symmetries are reconstructed from a $2n$ -reduction of the CKP hierarchy and its additional symmetries, which provides an alternative way to compute the obstacles in linearizing the Virasoro symmetries considered in [19]. Some remarks will be given in the final section.

2 The CKP hierarchy and its additional symmetries

Let \mathcal{A} be an algebra of smooth functions of a spatial coordinate x , and $D = d/dx$ be a derivation on \mathcal{A} . The algebra of pseudo-differential operators is the following linear space

$$\mathcal{D} = \left\{ \sum_{i < \infty} f_i D^i \mid f_i \in \mathcal{A} \right\} \quad (2.1)$$

equipped with a product defined by

$$f D^i \cdot g D^j = \sum_{r \geq 0} \binom{i}{r} f D^r(g) D^{i+j-r}, \quad f, g \in \mathcal{A}.$$

Given any operator $A = \sum_i f_i D^i \in \mathcal{D}$, its nonnegative part, negative part, residue and adjoint operator are respectively

$$A_+ = \sum_{i \geq 0} f_i D^i, \quad A_- = \sum_{i < 0} f_i D^i, \quad \text{res } A = f_{-1}, \quad A^* = \sum_i (-D)^i f_i. \quad (2.2)$$

These notions will be frequently used below.

Assume a pseudo-differential operator

$$L = D + \sum_{i \geq 1} v_i D^{-i} \in \mathcal{D} \quad (2.3)$$

satisfies $L^* = -L$. Note that each coefficient v_{2j} is a differential polynomial in the functions $v_1, v_3, \dots, v_{2j-1}$. The CKP hierarchy is defined by the following Lax equations:

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \quad k \in \mathbb{Z}_+^{\text{odd}}, \quad (2.4)$$

which form a system of evolutionary equations of the vector function $\mathbf{v} = (v_1, v_3, v_5, \dots)$ depending on $\mathbf{t} = (t_1, t_3, t_5, \dots)$. Clearly $\partial/\partial t_1 = \partial/\partial x$; henceforth we assume $t_1 = x$.

The operator L can be represented in a dressing form as

$$L = \Phi D \Phi^{-1}, \quad (2.5)$$

where Φ is a pseudo-differential operator:

$$\Phi = 1 + \sum_{i \geq 1} a_i D^{-i}, \quad \Phi^* = \Phi^{-1}. \quad (2.6)$$

Note that the dressing operator Φ is determined up to multiplication to the right by an arbitrary operator of the form (2.6) with constant coefficients. With the help of the dressing operator, the CKP hierarchy (2.4) can be redefined by the Sato equations:

$$\frac{\partial \Phi}{\partial t_k} = -(L^k)_- \Phi, \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (2.7)$$

Let $\xi(\mathbf{t}; z) = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} t_k z^k$ with some parameter z . Introduce a wave function

$$w(\mathbf{t}; z) = \Phi e^{\xi(\mathbf{t}; z)} = \phi(\mathbf{t}; z) e^{\xi(\mathbf{t}; z)}, \quad (2.8)$$

where

$$\phi(\mathbf{t}; z) = 1 + \sum_{i \geq 1} a_i z^{-i} \quad (2.9)$$

follows from the convention $D^i e^{xz} = z^i e^{xz}$ for any integer i . The dual wave function reads

$$w^*(\mathbf{t}; z) = (\Phi^{-1})^* e^{-\xi(\mathbf{t}; z)} = \phi(\mathbf{t}; -z) e^{-\xi(\mathbf{t}; z)} = w(\mathbf{t}; -z). \quad (2.10)$$

The CKP hierarchy (2.7), or (2.4), is equivalent to the following bilinear equation [2]:

$$\text{res}_z w(\mathbf{t}; z) w(\mathbf{t}'; -z) = 0. \quad (2.11)$$

Here $\text{res}_z \sum_i f_i z^i = f_{-1}$ for any formal Laurent series $\sum_i f_i z^i$ in z .

The additional symmetries for the CKP hierarchy were constructed by He, Tian, Foerster and Ma [8]. Their construction starts from introducing an Orlov-Schulman

[15] operator ¹

$$M = \Phi \Gamma \Phi^{-1}, \quad \Gamma = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} k t_k D^{k-1}.$$

Clearly $[L, M] = 1$. Given any pair of integers (m, l) with $m \geq 0$, let

$$A_{ml} = M^m L^l - (-1)^l L^l M^m. \quad (2.12)$$

In particular, one can check

$$A_{0l} = \begin{cases} 0, & l \text{ even}; \\ 2L^l, & l \text{ odd}, \end{cases} \quad (2.13)$$

$$A_{1l} = \begin{cases} -l L^{l-1}, & l \text{ even}; \\ 2M L^l + l L^{l-1}, & l \text{ odd}. \end{cases} \quad (2.14)$$

Note also $A_{ml}^* = -A_{ml}$, hence there are constants $c_{ml, m'l'}^{qr}$ such that

$$[A_{ml}, A_{m'l'}] = \sum_{q,r} c_{ml, m'l'}^{qr} A_{qr}.$$

In other words, all operators A_{ml} generate a centerless w_∞^C -algebra. As a matter of fact, only those A_{ml} with odd indices l are linearly independent, and the above structure constants are uniquely determined by letting $c_{ml, m'l'}^{qr} = 0$ for even r . For example,

$$c_{0l, 0l'}^{qr} = c_{0, l; 1, 2i}^{qr} = 0, \quad c_{0, 2i+1; 1, 2j+1}^{qr} = 2(2i+1)\delta_{q0}\delta_{r, 2(i+j)+1}, \quad (2.15)$$

$$c_{1, 2i+1; 1, 2j+1}^{qr} = 4(i-j)\delta_{q1}\delta_{r, 2(i+j)+1}. \quad (2.16)$$

The following equations are well defined

$$\frac{\partial \Phi}{\partial s_{ml}} = -(A_{ml})_- \Phi, \quad m \geq 0, \quad l \in \mathbb{Z}, \quad (2.17)$$

and these flows are assumed to commute with $\partial/\partial x$.

Proposition 2.1 ([8]) *The flows (2.17) commute with those in (2.7) that compose the CKP hierarchy. Moreover, the vector fields $\partial/\partial s_{ml}$ acting on the dressing operators Φ (or on the wave function $w(\mathbf{t}; z)$) satisfy*

$$\left[\frac{\partial}{\partial s_{ml}}, \frac{\partial}{\partial s_{m'l'}} \right] = - \sum_{q,r} c_{ml, m'l'}^{qr} \frac{\partial}{\partial s_{qr}}. \quad (2.18)$$

¹Strictly speaking, the operator M does not belong to the algebra \mathcal{D} in (2.1) for M may contain infinitely many terms of positive power in D . A trial to resolve this problem was given in [18], that is to assign certain degrees to t_k and extend \mathcal{D} to be the so-called algebra of pseudo-differential operators of the first type (cf. [14]). In this way L and M are contained in a common algebra so that the product between them makes sense.

This proposition means that, equations (2.17) define a set of symmetries, named as additional symmetries, for the CKP hierarchy. These additional symmetries acting on the wave function form a centerless w_∞^C -algebra.

Introduce a generating function of operators

$$Y(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l-1} (A_{m,m+l})_- \quad (2.19)$$

with parameters λ and μ . He et al [8] obtained the following

Proposition 2.2 *The generator (2.19) can be represented as*

$$Y(\lambda, \mu) = w(\mathbf{t}; -\lambda) D^{-1} w(\mathbf{t}; \mu) + w(\mathbf{t}; \mu) D^{-1} w(\mathbf{t}; -\lambda). \quad (2.20)$$

3 Tau function of the CKP hierarchy

We are to introduce a tau function of the CKP hierarchy. To this end let us first rewrite the hierarchy into the form of Hamiltonian systems.

Given an arbitrary positive integer n , the operator L in (2.3) satisfies $(L^{2n})^* = L^{2n}$. Assume F to be a formal functional depending on L :

$$F = \int f(\mathbf{v}, \partial_x \mathbf{v}, \partial_x^2 \mathbf{v}, \dots) dx \in \mathcal{A}/\partial_x \mathcal{A}. \quad (3.1)$$

Its variational derivative with respect to L^{2n} is defined to be a pseudo-differential operator P such that

$$\delta F = \int \text{res}(P \delta L^{2n}) dx, \quad P^* = -P.$$

Let F_P denote the functional whose variational derivative with respect to L^{2n} is $P \in \mathcal{D}$. For such functionals, there is a pair of compatible Poisson brackets that are reduced from those in the bi-Hamiltonian representations of the KP hierarchy (see [5] and references therein):

$$\{F_P, F_Q\}_1^n = \int \text{res}(P([-Q_-], L^{2n}] + [Q, L^{2n}]_-)) dx, \quad (3.2)$$

$$\{F_P, F_Q\}_2^n = \int \text{res}(P(-(L^{2n}Q)_- L^{2n} + L^{2n}(Q L^{2n})_-)) dx. \quad (3.3)$$

Then the CKP hierarchy (2.4) can be represented in a bi-Hamiltonian recursive form as

$$\frac{\partial F}{\partial t_k} = \{F, H_{k+2n}\}_1^n = \{F, H_k\}_2^n, \quad k \in \mathbb{Z}_+^{\text{odd}}, \quad (3.4)$$

where F is an arbitrary functional of the form (3.1), and the Hamiltonians are

$$H_k = \frac{2n}{k} \int \text{res} L^k dx. \quad (3.5)$$

The Hamiltonian densities in (3.5) are tau-symmetric [7]. That is to say, they define a closed 1-form

$$\omega = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \text{res } L^k dt_k,$$

hence there locally is a smooth function $\tau(\mathbf{t})$ such that

$$d(2 \partial_x \log \tau) = \omega. \quad (3.6)$$

More precisely, we have

Definition 3.1 The smooth function $\tau(\mathbf{t})$ satisfying

$$\frac{\partial^2 \log \tau}{\partial t_k \partial t_l} = \frac{1}{2} \partial_x^{-1} \text{res}[(L^k)_+, L^l], \quad k, l \in \mathbb{Z}_+^{\text{odd}} \quad (3.7)$$

is called a tau function of the CKP hierarchy (2.4). Here on the right hand side of (3.7) the integration constants are taken to be zero (the residue of any commutator of pseudo-differential operators is a total derivative in x).

Observe that $\log \tau$ is determined up to addition of a linear function of the time variables.

In order to relate the tau function to the wave function (2.8) of the CKP hierarchy, we introduce the following shift operator

$$G(\mathbf{t}; z) = \exp \left(- \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{2}{k} z^k \frac{\partial}{\partial t_k} \right).$$

We also write $G(z) = G(\mathbf{t}; z)$ in case no confusion would happen.

Proposition 3.2 *For the CKP hierarchy, the tau function and the wave function are related via the following formula*

$$w(\mathbf{t}; z) = \left(1 + \frac{1}{z} \partial_x \log \frac{G(z) \tau(\mathbf{t})}{\tau(\mathbf{t})} \right)^{1/2} \frac{G(z) \tau(\mathbf{t})}{\tau(\mathbf{t})} e^{\xi(\mathbf{t}; z)}. \quad (3.8)$$

To prove this proposition, we need the following two lemmas.

Lemma 3.3 *Recall the dressing operator (2.6) for the CKP hierarchy. It holds that*

$$a_1(\mathbf{t}) = -2 \partial_x \log \tau(\mathbf{t}). \quad (3.9)$$

Proof The lemma follows from taking the residue of equation (2.7) and the definition of the tau function. \square

Lemma 3.4 *Let $\varphi(\mathbf{t}; \lambda) = \phi(\mathbf{t}; \lambda) G(\lambda) \phi(\mathbf{t}; -\lambda)$ (recall ϕ in (2.9)) with λ being a parameter. Then it satisfies*

(i)

$$\varphi(\mathbf{t}; \lambda) = 1 + \frac{1}{2\lambda}(1 - G(\lambda))a_1(\mathbf{t}); \quad (3.10)$$

(ii)

$$2 \partial_x \log \phi(\mathbf{t}; \lambda) - \partial_x \log \varphi(\mathbf{t}; \lambda) = (1 - G(\lambda))a_1(\mathbf{t}). \quad (3.11)$$

Proof According to the bilinear equation (2.11), we have

$$\begin{aligned} 0 &= \text{res}_z \phi(\mathbf{t}; z) e^{\xi(\mathbf{t}; z)} G(\lambda) (\phi(\mathbf{t}; -z) e^{-\xi(\mathbf{t}; z)}) \\ &= \text{res}_z \phi(\mathbf{t}; z) G(\lambda) \phi(\mathbf{t}; -z) \frac{1 + z/\lambda}{1 - z/\lambda} \\ &= \lambda \left(\phi(\mathbf{t}; z) G(\lambda) \phi(\mathbf{t}; -z) \left(1 + \frac{z}{\lambda} \right) \right) \Big|_{z=\lambda}^- \\ &= \lambda \left(\phi(\mathbf{t}; z) G(\lambda) \phi(\mathbf{t}; -z) \left(1 + \frac{z}{\lambda} \right) - 1 - (1 + a_1(\mathbf{t})z^{-1} - G(\lambda)a_1(\mathbf{t})z^{-1}) \frac{z}{\lambda} \right) \Big|_{z=\lambda} \\ &= 2\lambda \phi(\mathbf{t}; \lambda) G(\lambda) \phi(\mathbf{t}; -\lambda) - 2\lambda - (1 - G(\lambda))a_1(\mathbf{t}), \end{aligned} \quad (3.12)$$

in the third equality of which the subscript “ $-$ ” means to take the negative-power part of a series in z . Thus the first formula (3.10) is valid.

Secondly, the bilinear equation (2.11) also yields

$$\begin{aligned} 0 &= \text{res}_z (\partial_x (\phi(\mathbf{t}; z) e^{\xi(\mathbf{t}; z)})) G(\lambda) (\phi(\mathbf{t}; -z) e^{-\xi(\mathbf{t}; z)}) \\ &= \text{res}_z (z \phi(\mathbf{t}; z) + \partial_x \phi(\mathbf{t}; z)) G(\lambda) \phi(\mathbf{t}; -z) \frac{1 + z/\lambda}{1 - z/\lambda} \\ &= \lambda \left((z \phi(\mathbf{t}; z) + \partial_x \phi(\mathbf{t}; z)) G(\lambda) \phi(\mathbf{t}; -z) \left(1 + \frac{z}{\lambda} \right) \right) \Big|_{z=\lambda}^- \\ &= 2\lambda (\lambda \phi(\mathbf{t}; \lambda) + \partial_x \phi(\mathbf{t}; \lambda)) G(\lambda) \phi(\mathbf{t}; -\lambda) \\ &\quad - \lambda \cdot \lambda (1 + a_1(\mathbf{t})\lambda^{-1} - G(\lambda)a_1(\mathbf{t})\lambda^{-1}) \\ &\quad - \lambda \cdot \lambda (1 + a_1(\mathbf{t})\lambda^{-1} + a_2(\mathbf{t})\lambda^{-2} - G(\lambda)a_1(\mathbf{t})\lambda^{-1} + G(\lambda)a_2(\mathbf{t})\lambda^{-2} \\ &\quad - a_1(\mathbf{t})G(\lambda)a_1(\mathbf{t})\lambda^{-2}) - \lambda \partial_x a_1(\mathbf{t})\lambda^{-1} \\ &= 2\lambda^2 \varphi(\mathbf{t}; \lambda) + 2\lambda \partial_x \phi(\mathbf{t}; \lambda) \cdot G(\lambda) \phi(\mathbf{t}; -\lambda) - 2\lambda^2 \\ &\quad - 2\lambda(1 - G(\lambda))a_1(\mathbf{t}) - (1 + G(\lambda))a_2(\mathbf{t}) + a_1(\mathbf{t})G(\lambda)a_1(\mathbf{t}) - \partial_x a_1(\mathbf{t}). \end{aligned} \quad (3.13)$$

On the other hand, it follows from (2.6) that $2a_2(\mathbf{t}) = a_1(\mathbf{t})^2 - \partial_x a_1(\mathbf{t})$, hence

$$\begin{aligned} &2(1 + G(\lambda))a_2(\mathbf{t}) \\ &= ((1 - G(\lambda))a_1(\mathbf{t}))^2 + 2a_1(\mathbf{t})G(\lambda)a_1(\mathbf{t}) - (1 + G(\lambda))\partial_x a_1(\mathbf{t}). \end{aligned} \quad (3.14)$$

Substituting (3.10) and (3.14) into (3.13), one deduces

$$\begin{aligned} &2\lambda \partial_x \phi(\mathbf{t}; \lambda) \cdot G(\lambda) \phi(\mathbf{t}; -\lambda) \\ &= \lambda(1 - G(\lambda))a_1(\mathbf{t}) + \frac{1}{2}((1 - G(\lambda))a_1(\mathbf{t}))^2 + \frac{1}{2}(1 - G(\lambda))\partial_x a_1(\mathbf{t}) \end{aligned}$$

$$= \lambda \varphi(\mathbf{t}; \lambda) (1 - G(\lambda)) a_1(\mathbf{t}) + \lambda \partial_x \varphi(\mathbf{t}; \lambda). \quad (3.15)$$

Divide both sides by $\lambda \varphi(\mathbf{t}; \lambda)$, then we obtain (3.11). The lemma is proved. \square

Proof of Proposition 3.2 Substituting (3.9) into (3.10) and (3.11), we have respectively

$$\varphi(\mathbf{t}; \lambda) = 1 + \frac{1}{\lambda} (G(\lambda) - 1) \partial_x \log \tau(\mathbf{t}) = 1 + \frac{1}{\lambda} \partial_x \log \frac{G(\lambda) \tau(\mathbf{t})}{\tau(\mathbf{t})}, \quad (3.16)$$

$$\phi(\mathbf{t}; \lambda) = \sqrt{\varphi(\mathbf{t}; \lambda)} \frac{G(\lambda) \tau(\mathbf{t})}{\tau(\mathbf{t})}. \quad (3.17)$$

They, with λ replaced by z , lead to the equality (3.8) by virtue of the definition of the wave function (2.8). Therefore the proposition is proved. \square

Denote

$$X(\mathbf{t}; z) = e^{\xi(\mathbf{t}; z)} G(\mathbf{t}; z). \quad (3.18)$$

Now we achieve the main result of the present section.

Theorem 3.5 *The CKP hierarchy (2.4) is equivalent to the following “bilinear” equation of tau function:*

$$\text{res}_z \sqrt{\varphi(\mathbf{t}; z) \varphi(\mathbf{t}'; -z)} X(\mathbf{t}; z) \tau(\mathbf{t}) X(\mathbf{t}'; z) \tau(\mathbf{t}') = 0, \quad (3.19)$$

where the function $\varphi(\mathbf{t}; z)$ is given in (3.16).

Observe the difference between (3.19) and those Hirota bilinear equations of the usual sense (for example, the KP and the BKP hierarchies [3]) in the literature: now there is a square-root factor given by the tau function!

At the end of this section, we note that $\varphi(\mathbf{t}; z) = 1 + O(1/z^2)$ as $z \rightarrow \infty$, and that it satisfies

$$G(-z) \varphi(\mathbf{t}; z) = \varphi(\mathbf{t}; -z). \quad (3.20)$$

These properties will be employed in the forthcoming section.

4 Additional symmetries represented via tau function

In this section we want to represent the additional symmetries (2.17) for the CKP hierarchy via the tau function introduced above. Our main tool is served by vertex operators.

For $k \in \mathbb{Z}_+^{\text{odd}}$, denote

$$p_k = 2 \frac{\partial}{\partial t_k}, \quad p_{-k} = k t_k.$$

Clearly $[p_k, p_l] = 2k \delta_{k, -l}$. Introduce a vertex operator

$$X(\mathbf{t}; \lambda, \mu) = : \exp \left(\sum_{k \in \mathbb{Z}^{\text{odd}}} \frac{p_k}{k \lambda^k} - \sum_{k \in \mathbb{Z}^{\text{odd}}} \frac{p_k}{k \mu^k} \right) :, \quad (4.1)$$

where the normal-order product “:” means to place $p_{k>0}$ to the right of $p_{k<0}$, and λ and μ are parameters. Without any confusion we will simply write $X(\lambda, \mu) = X(\mathbf{t}; \lambda, \mu)$.

Lemma 4.1 *The tau function in (3.7) of the CKP hierarchy satisfies the following equality*

$$\begin{aligned} & \partial_x \frac{X(\lambda, \mu) \tau(\mathbf{t})}{\tau(\mathbf{t})} \\ &= \frac{\mu - \lambda}{\mu + \lambda} w(\mathbf{t}; \mu) w(\mathbf{t}; -\lambda) \left(\lambda G(\mu) \sqrt{\frac{\varphi(\mathbf{t}; -\lambda)}{\varphi(\mathbf{t}; -\mu)}} + \mu G(-\lambda) \sqrt{\frac{\varphi(\mathbf{t}; \mu)}{\varphi(\mathbf{t}; \lambda)}} \right), \end{aligned} \quad (4.2)$$

where the functions $w(\mathbf{t}; z)$ and $\varphi(\mathbf{t}; z)$ are given by (3.8) and (3.16) respectively.

Proof We write

$$X(\lambda, \mu) = X(\mathbf{t}; \lambda, \mu) = e^{-\xi(\mathbf{t}; \lambda) + \xi(\mathbf{t}; \mu)} G(-\lambda) G(\mu).$$

Recall (3.18). It is straightforward to calculate

$$\begin{aligned} X(\mathbf{t}; z) X(\mathbf{t}; \lambda, \mu) &= \frac{1 - \mu/z}{1 + \mu/z} \frac{1 + \lambda/z}{1 - \lambda/z} e^{\xi(\mathbf{t}; z) - \xi(\mathbf{t}; \lambda) + \xi(\mathbf{t}; \mu)} G(z) G(-\lambda) G(\mu), \\ X(\mathbf{t}; \lambda, \mu) X(\mathbf{t}; z) &= \frac{1 - z/\mu}{1 + z/\mu} \frac{1 + z/\lambda}{1 - z/\lambda} e^{\xi(\mathbf{t}; z) - \xi(\mathbf{t}; \lambda) + \xi(\mathbf{t}; \mu)} G(z) G(-\lambda) G(\mu). \end{aligned}$$

The bilinear equation (3.19) yields

$$\begin{aligned} 0 &= \text{res}_z X(\mathbf{t}; \lambda, \mu) \sqrt{\varphi(\mathbf{t}; z) \varphi(\mathbf{t}'; -z)} X(\mathbf{t}; z) \tau(\mathbf{t}) X(\mathbf{t}'; -z) \tau(\mathbf{t}') \\ &= \text{res}_z \sqrt{G(-\lambda) G(\mu) \varphi(\mathbf{t}; z) \cdot \varphi(\mathbf{t}'; -z)} X(\mathbf{t}; \lambda, \mu) X(\mathbf{t}; z) \tau(\mathbf{t}) X(\mathbf{t}'; -z) \tau(\mathbf{t}') \\ &= \text{res}_z \sqrt{G(-\lambda) G(\mu) \varphi(\mathbf{t}; z) \cdot \varphi(\mathbf{t}'; -z)} X(\mathbf{t}; z) X(\mathbf{t}; \lambda, \mu) \tau(\mathbf{t}) X(\mathbf{t}'; -z) \tau(\mathbf{t}') \\ &\quad - \text{res}_z \left(a(z, \lambda, \mu) e^{-\xi(\mathbf{t}; \lambda) + \xi(\mathbf{t}; \mu)} \sqrt{G(-\lambda) G(\mu) \varphi(\mathbf{t}; z) \cdot \varphi(\mathbf{t}'; -z)} \times \right. \\ &\quad \left. \times X(\mathbf{t}; z) G(-\lambda) G(\mu) \tau(\mathbf{t}) X(\mathbf{t}'; -z) \tau(\mathbf{t}') \right) \end{aligned} \quad (4.3)$$

with

$$a(z, \lambda, \mu) = \frac{1 - \mu/z}{1 + \mu/z} \frac{1 + \lambda/z}{1 - \lambda/z} - \frac{1 - z/\mu}{1 + z/\mu} \frac{1 + z/\lambda}{1 - z/\lambda}.$$

The function $a(z, \lambda, \mu)$ can be rewritten as

$$a(z, \lambda, \mu) = \frac{1 + \lambda/z}{1 + \mu/z} (z - \mu) \delta(z, \lambda) + \frac{\mu}{\lambda} \frac{1 - z/\mu}{1 - z/\lambda} (z + \lambda) \delta(z, -\mu), \quad (4.4)$$

where $\delta(z, \lambda) = (z(1 - \lambda/z))^{-1} + (\lambda(1 - z/\lambda))^{-1}$ satisfies $\text{res}_z f(z) \delta(z, \lambda) = f(\lambda)$ for any Laurent series $f(z)$, see [5]. Thus by taking $\mathbf{t}' = \mathbf{t}$ in (4.3) and using (3.20) we have

$$\text{res}_z \sqrt{G(-\lambda) G(\mu) \varphi(\mathbf{t}; z) \cdot \varphi(\mathbf{t}; -z)} \frac{G(z) X(\lambda, \mu) \tau(\mathbf{t})}{\tau(\mathbf{t})} \frac{G(-z) \tau(\mathbf{t})}{\tau(\mathbf{t})}$$

$$\begin{aligned}
&= \frac{2(\lambda - \mu)}{1 + \mu/\lambda} e^{-\xi(\mathbf{t}; \lambda) + \xi(\mathbf{t}; \mu)} \sqrt{G(-\lambda)G(\mu)\varphi(\mathbf{t}; \lambda) \cdot \varphi(\mathbf{t}; -\lambda)} \frac{G(\mu)\tau(\mathbf{t})}{\tau(\mathbf{t})} \frac{G(-\lambda)\tau(\mathbf{t})}{\tau(\mathbf{t})} \\
&\quad + \frac{\mu}{\lambda} \frac{2(\lambda - \mu)}{1 + \mu/\lambda} e^{-\xi(\mathbf{t}; \lambda) + \xi(\mathbf{t}; \mu)} \sqrt{G(-\lambda)G(\mu)\varphi(\mathbf{t}; -\mu) \cdot \varphi(\mathbf{t}; \mu)} \frac{G(-\lambda)\tau(\mathbf{t})}{\tau(\mathbf{t})} \frac{G(\mu)\tau(\mathbf{t})}{\tau(\mathbf{t})} \\
&= \frac{2\lambda(\lambda - \mu)}{\lambda + \mu} \sqrt{\frac{G(\mu)\varphi(\mathbf{t}; -\lambda)}{\varphi(\mathbf{t}; \mu)}} w(\mathbf{t}; \mu) w(\mathbf{t}; -\lambda) \\
&\quad + \frac{2\mu(\lambda - \mu)}{\lambda + \mu} \sqrt{\frac{G(-\lambda)\varphi(\mathbf{t}; \mu)}{\varphi(\mathbf{t}; -\lambda)}} w(\mathbf{t}; \mu) w(\mathbf{t}; -\lambda) \\
&= \frac{2(\lambda - \mu)}{\mu + \lambda} w(\mathbf{t}; \mu) w(\mathbf{t}; -\lambda) \left(\lambda G(\mu) \sqrt{\frac{\varphi(\mathbf{t}; -\lambda)}{\varphi(\mathbf{t}; -\mu)}} + \mu G(-\lambda) \sqrt{\frac{\varphi(\mathbf{t}; \mu)}{\varphi(\mathbf{t}; \lambda)}} \right). \tag{4.5}
\end{aligned}$$

Recall $\varphi(\mathbf{t}; z) = 1 + O(1/z^2)$, hence the left-hand side of this equation is

$$\begin{aligned}
\text{l.h.s.} &= \text{res}_z \frac{G(z)X(\lambda, \mu)\tau(\mathbf{t})}{\tau(\mathbf{t})} \frac{G(-z)\tau(\mathbf{t})}{\tau(\mathbf{t})} \\
&= \frac{-2\partial_x(X(\lambda, \mu)\tau(\mathbf{t}))}{\tau(\mathbf{t})} + \frac{X(\lambda, \mu)\tau(\mathbf{t})}{\tau(\mathbf{t})} \frac{2\partial_x\tau(\mathbf{t})}{\tau(\mathbf{t})} \\
&= -2\partial_x \frac{X(\lambda, \mu)\tau(\mathbf{t})}{\tau(\mathbf{t})}. \tag{4.6}
\end{aligned}$$

Substitute it into (4.5), then we obtain (4.10). The lemma is proved. \square

One expands the vertex operator (4.1) formally as

$$X(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l} W_l^{(m)} \tag{4.7}$$

with

$$W_l^{(m)} = \text{res}_\lambda (\lambda^{m+l-1} \partial_\mu^m |_{\mu=\lambda} X(\lambda, \mu)).$$

For convenience we assume $\dot{p}_i = 0$ for even i . It is straightforward to calculate

$$\begin{aligned}
W_l^{(0)} &= \delta_{l0}, \quad W_l^{(1)} = p_l, \quad W_l^{(2)} = \sum_{i+j=l} : p_i p_j : - (l+1) p_l, \\
W_l^{(3)} &= \sum_{i+j+k=l} : p_i p_j p_k : - \frac{3}{2} (l+2) \sum_{i+j=l} : p_i p_j : + (l+2)(l+1) p_l.
\end{aligned}$$

Proposition 4.2 *For the CKP hierarchy, the additional symmetries (2.17) acting on the tau function are given by the following formula*

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l-1} \partial_x \left(\frac{1}{\tau(\mathbf{t})} \frac{\partial \tau(\mathbf{t})}{\partial s_{m,m+l}} - \frac{1}{m+1} \frac{W_l^{(m+1)} \tau(\mathbf{t})}{\tau(\mathbf{t})} \right) \\
&= w(\mathbf{t}; \mu) w(\mathbf{t}; -\lambda) \left(1 - \frac{\lambda}{\mu + \lambda} G(\mu) \sqrt{\frac{\varphi(\mathbf{t}; -\lambda)}{\varphi(\mathbf{t}; -\mu)}} - \frac{\mu}{\mu + \lambda} G(-\lambda) \sqrt{\frac{\varphi(\mathbf{t}; \mu)}{\varphi(\mathbf{t}; \lambda)}} \right). \tag{4.8}
\end{aligned}$$

Proof Let

$$Z(\lambda, \mu) = \frac{1}{\mu - \lambda} (X(\lambda, \mu) - 1) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l-1} \frac{W_l^{(m+1)}}{m+1}. \quad (4.9)$$

Lemma 4.1 implies

$$\begin{aligned} & \partial_x \frac{Z(\lambda, \mu) \tau(\mathbf{t})}{\tau(\mathbf{t})} \\ &= w(\mathbf{t}; \mu) w(\mathbf{t}; -\lambda) \left(\frac{\lambda}{\mu + \lambda} G(\mu) \sqrt{\frac{\varphi(\mathbf{t}; -\lambda)}{\varphi(\mathbf{t}; -\mu)}} + \frac{\mu}{\mu + \lambda} G(-\lambda) \sqrt{\frac{\varphi(\mathbf{t}; \mu)}{\varphi(\mathbf{t}; \lambda)}} \right). \end{aligned} \quad (4.10)$$

Hence to show (4.8) we only need to verify

$$\sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l-1} \partial_x \frac{\partial \log \tau(\mathbf{t})}{\partial s_{m,m+l}} = w(\mathbf{t}; \mu) w(\mathbf{t}; -\lambda). \quad (4.11)$$

In fact, according to (2.17) and (3.9) one has

$$\text{res} A_{m,m+l} = -\frac{\partial a_1}{\partial s_{m,m+l}} = 2 \partial_x \frac{\partial \log \tau(\mathbf{t})}{\partial s_{m,m+l}}.$$

Thus the equality (4.11) follows from taking the residue of (2.20). Therefore the proposition is proved. \square

Corollary 4.3 *The additional symmetries (2.17) with $m = 0$ and 1 can be represented as follows*

(i) for $l \in \mathbb{Z}$,

$$\frac{\partial \tau}{\partial s_{0l}} = \left(W_l^{(1)} + \delta_{l0} c^{(1)} \right) \tau; \quad (4.12)$$

(ii) for $i \in \mathbb{Z}$,

$$\frac{\partial \tau}{\partial s_{1,2i}} = \frac{1}{2} W_{2i-1}^{(2)} \tau; \quad (4.13)$$

$$\frac{\partial \tau}{\partial s_{1,2i+1}} - \left(\frac{1}{2} W_{2i}^{(2)} + \delta_{i0} c^{(2)} \right) \tau = \begin{cases} 0, & i \leq 0; \\ \tau \partial_x^{-1} T_{2i+1}, & i \geq 1, \end{cases} \quad (4.14)$$

in which

$$T_{2i+1} = -\frac{1}{2} \text{res}_\lambda \lambda^{2i+1} \frac{G(\lambda) \tau}{\tau} \frac{G(-\lambda) \tau}{\tau} \sqrt{\frac{\varphi(\mathbf{t}; -\lambda)}{\varphi(\mathbf{t}; \lambda)}} (N(\lambda) - \partial_\lambda) \varphi(\mathbf{t}; \lambda) \quad (4.15)$$

with

$$N(\lambda) = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{2}{\lambda^{k+1}} \frac{\partial}{\partial t_k}.$$

Here $c^{(1)}$ and $c^{(2)}$ are certain constants that arise from a central extension of the w_∞^C -algebra.

Proof Clearly the right-hand side of (4.8) vanishes whenever $\mu = \lambda$, which implies the first assertion. Let us proceed to show the second one.

It is easy to see

$$[\partial_\lambda, G(\lambda)] = N(\lambda)G(\lambda), \quad [\partial_\lambda, G(-\lambda)] = -N(\lambda)G(-\lambda).$$

Denote

$$\chi(\mathbf{t}; \lambda, \mu) = 1 - \frac{\lambda}{\mu + \lambda} G(\mu) \sqrt{\frac{\varphi(\mathbf{t}; -\lambda)}{\varphi(\mathbf{t}; -\mu)}} - \frac{\mu}{\mu + \lambda} G(-\lambda) \sqrt{\frac{\varphi(\mathbf{t}; \mu)}{\varphi(\mathbf{t}; \lambda)}}. \quad (4.16)$$

One has $\chi(\mathbf{t}; \lambda, \lambda) = 0$ and

$$\begin{aligned} & \partial_\mu|_{\mu=\lambda} \chi(\mathbf{t}; \lambda, \mu) \\ &= \frac{1}{4} G(\lambda) \partial_\lambda \log \varphi(\mathbf{t}; -\lambda) - \frac{1}{4} G(-\lambda) \partial_\lambda \log \varphi(\mathbf{t}; \lambda) \\ &= \frac{1}{4} (-N(\lambda) + \partial_\lambda) \log \varphi(\mathbf{t}; \lambda) - \frac{1}{4} (N(\lambda) + \partial_\lambda) \log \varphi(\mathbf{t}; -\lambda) \\ &= -\frac{1}{4} N(\lambda) \log (\varphi(\mathbf{t}; \lambda) \varphi(\mathbf{t}; -\lambda)) + \frac{1}{4} \partial_\lambda \log \frac{\varphi(\mathbf{t}; \lambda)}{\varphi(\mathbf{t}; -\lambda)}. \end{aligned} \quad (4.17)$$

Hence

$$\begin{aligned} & \partial_x \left(\frac{1}{\tau(\mathbf{t})} \frac{\partial \tau(\mathbf{t})}{\partial s_{1,l+1}} - \frac{1}{2} \frac{W_l^{(2)} \tau(\mathbf{t})}{\tau(\mathbf{t})} \right) \\ &= \text{res}_\lambda \lambda^{l+1} \partial_\mu|_{\mu=\lambda} (w(\mathbf{t}; \mu) w(\mathbf{t}; -\lambda) \chi(\mathbf{t}; \lambda, \mu)) \\ &= -\frac{1}{4} \text{res}_\lambda \lambda^{l+1} w(\mathbf{t}; \lambda) w(\mathbf{t}; -\lambda) \left(N(\lambda) \log (\varphi(\mathbf{t}; \lambda) \varphi(\mathbf{t}; -\lambda)) - \partial_\lambda \log \frac{\varphi(\mathbf{t}; \lambda)}{\varphi(\mathbf{t}; -\lambda)} \right). \end{aligned} \quad (4.18)$$

Since (4.17) is an even function in λ , then (4.18) vanishes whenever l is odd. Namely, the equality (4.13) is verified. When $l = 2i$ is even, we substitute (2.8) and (3.17) into the right-hand side of (4.18), then it is recast to T_{2i+1} after a straight forward calculation. The corollary is proved. \square

Example 4.4 Let us illustrate how to compute the tails T_{2i+1} in (4.14). For this purpose, we expand

$$G(\lambda) = 1 + \sum_{j \geq 1} \sigma_j(-\tilde{\boldsymbol{\theta}}) \frac{1}{\lambda^j}, \quad \tilde{\boldsymbol{\theta}} = \left(\frac{2}{1} \frac{\partial}{\partial t_1}, 0, \frac{2}{3} \frac{\partial}{\partial t_3}, 0, \dots \right),$$

where σ_j are polynomials defined by $\exp(\sum_{i \geq 1} q_i z^i) = \sum_{j \geq 0} \sigma_j(q_1, q_2, \dots) z^j$. Denote $\theta = \log \tau$, and $\theta_x = \partial_x \theta$, $\theta_{x t_3} = \partial_x \partial_{t_3} \theta$ etc. (recall $x = t_1$). One has

$$\varphi(\mathbf{t}; \lambda) = 1 + \frac{1}{\lambda} (G(\lambda) - 1) \theta_x = 1 + \sum_{j \geq 1} \frac{1}{\lambda^{j+1}} \sigma_j(-\tilde{\boldsymbol{\theta}}) \theta_x = 1 + O(1/\lambda^2), \quad (4.19)$$

$$\begin{aligned}
& (N(\lambda) - \partial_\lambda) \varphi(\mathbf{t}; \lambda) \\
&= \frac{1}{\lambda^2} (G(\lambda) - 1) \theta_x - \frac{1}{\lambda} N(\lambda) \theta_x \\
&= \sum_{i \geq 1} \left(\frac{1}{\lambda^{2i+1}} \left(\sigma_{2i-1}(-\tilde{\boldsymbol{\theta}}) - 2 \frac{\partial}{\partial t_{2i-1}} \right) \theta_x + \frac{1}{\lambda^{2i+2}} \sigma_{2i}(-\tilde{\boldsymbol{\theta}}) \theta_x \right) = O(1/\lambda^3), \quad (4.20)
\end{aligned}$$

$$\begin{aligned}
& \frac{G(\lambda) \tau}{\tau} \frac{G(-\lambda) \tau}{\tau} \\
&= \exp((G(\lambda) + G(-\lambda) - 2)\theta) \\
&= \exp\left(2 \sum_{i \geq 1} \frac{1}{\lambda^{2i}} \sigma_{2i}(-\tilde{\boldsymbol{\theta}}) \theta\right) = 1 + O(1/\lambda^2). \quad (4.21)
\end{aligned}$$

Substitute them into (4.15), then it is straight forward to obtain

$$T_3 = -\theta_{xxx}, \quad T_5 = -\partial_x \left(\frac{2}{3} \theta_{xt_3} + \frac{1}{3} \theta_{xxxx} + 4\theta_{xx}^2 \right). \quad (4.22)$$

Generally, for $i \geq 1$,

$$T_{2i+1} = -\frac{1}{2} \sigma_{2i}(-\tilde{\boldsymbol{\theta}}) \theta_x + \text{nonlinear terms in derivatives of } \theta. \quad (4.23)$$

Note that the nonlinear terms are trivial whenever $i = 1$.

Claim For every $i \geq 1$, it holds that $T_{2i+1} = \partial_x \tilde{T}_{2i}$ for some polynomial

$$\tilde{T}_{2i} \in \mathbb{C} \left[\frac{\partial^{m_1+\dots+m_s}}{\partial t_{k_1}^{m_1} \dots \partial t_{k_s}^{m_s}} \theta; \ m_1 + \dots + m_s \geq 2 \right]. \quad (4.24)$$

Moreover, each polynomial \tilde{T}_{2i} is homogeneous of degree $2i$ if we assign

$$\deg \frac{\partial^{m_1+\dots+m_s}}{\partial t_{k_1}^{m_1} \dots \partial t_{k_s}^{m_s}} \theta = m_1 k_1 + \dots + m_s k_s.$$

The validity of the claim will be verified below, though not in so direct a way.

With the same method as in Corollary 4.3, from (4.8) one can calculate $\partial \tau / \partial s_{m,m+l}$ for $m \geq 2$. However, the formulae turn out complicated.

Now we conclude that, for the CKP hierarchy, when lifting the actions of additional symmetries on the wave function to the actions on the tau function one has not only a central extension of the w_∞^C -algebra but also certain nontrivial “tails” like given by T_{2i+1} . This is the main difference between the additional symmetries for the CKP hierarchy and those as for the KP hierarchy considered before [1, 4, 12, 16, 18].

5 Virasoro symmetries for Drinfeld-Sokolov hierarchies of type C

In this section we want to study the Virasoro symmetries for Drinfeld-Sokolov hierarchies of type C. For this purpose, we first recall the definition of these Drinfeld-Sokolov hierarchies and their Virasoro symmetries [6, 19], then reconstruct them starting from certain reductions of the CKP hierarchy based on the above results.

5.1 Drinfeld-Sokolov hierarchies of type C

Instead of getting into Drinfeld and Sokolov's general construction [6], let us recall only the definition of the hierarchy associated to affine Kac-Moody algebra \mathfrak{g} of type $C_n^{(1)}$ with the marked vertex being the zeroth one, i.e., the special vertex added to the Dynkin diagram of the corresponding simple Lie algebra $\mathring{\mathfrak{g}}$ of type C_n .

For any integer $n \geq 2$ fixed, one realizes \mathfrak{g} as $\mathfrak{sp}(2n)[\lambda, 1/\lambda] \oplus \mathbb{C}c \oplus \mathbb{C}d$ with c and d being the canonical central element and the scaling element respectively. In more details, let $e_{i,j}$ be the $2n \times 2n$ matrix whose (i, j) -entry takes value 1 and the other entries vanish, then a set of Weyl generators of \mathfrak{g} can be chosen as follows [6, 10]:

$$e_i = e_{i+1,i} + e_{2n-i+1,2n-i} \quad (1 \leq i \leq n-1), \quad (5.1)$$

$$e_n = e_{n+1,n}, \quad e_0 = \lambda e_{1,2n}, \quad (5.2)$$

$$f_i = e_{i,i+1} + e_{2n-i,2n-i+1} \quad (1 \leq i \leq n-1), \quad (5.3)$$

$$f_n = e_{n,n+1}, \quad f_0 = \lambda^{-1} e_{2n,1}, \quad (5.4)$$

$$\alpha_i^\vee = [e_i, f_i] \quad (1 \leq i \leq n), \quad (5.5)$$

$$\alpha_0^\vee = e_{1,1} - e_{2n,2n} + c. \quad (5.6)$$

Denote $\Lambda = \sum_{i=0}^n e_i$. The elements Λ^k with $k \in \mathbb{Z}^{\text{odd}}$ generate the principal Heisenberg subalgebra of \mathfrak{g} . Moreover, they satisfy

$$(\Lambda^k | \Lambda^l) = 2n \delta_{k,l}, \quad k, l \in \mathbb{Z}^{\text{odd}}$$

for the standard invariant bilinear (Killing) form $(\cdot | \cdot)$ on \mathfrak{g} . Note that $2n$ is the Coxeter number.

Introduce a matrix operator

$$\mathcal{L} = D + \Lambda + q \quad (5.7)$$

with $D = d/dx$ and q being a smooth function of x that takes value in the Borel subalgebra of $\mathring{\mathfrak{g}}$ generated by α_i^\vee and f_i with $1 \leq i \leq n$. The nilpotent subalgebra, say \mathfrak{n} , generated by f_i with $1 \leq i \leq n$, induces a group of gauge transformations of \mathcal{L} as

$$\mathcal{L} \mapsto e^{\text{ad}_N} \mathcal{L}, \quad N \in \mathfrak{n}. \quad (5.8)$$

The Drinfeld-Sokolov hierarchy associated to \mathfrak{g} is defined to be

$$\frac{\partial \mathcal{L}}{\partial t_k} = [\mathcal{A}(\Lambda^k), \mathcal{L}], \quad k \in E_+ \quad (5.9)$$

modulo the gauge transformations (5.8). Here $\mathcal{A}(\Lambda^k)$, depending on Λ^k , are certain \mathfrak{g} -valued differential polynomials in q , see [6] (also [19]) for details.

For the equivalence class of \mathcal{L} with respect to the transformations (5.8), a representative element can be chosen as

$$\mathcal{L}^{\text{can}} = D + \Lambda + q^{\text{can}}, \quad q^{\text{can}} = - \sum_{i=1}^n \frac{u_i}{2} (e_{1,2i} + e_{2n-2i+1,2n}) \quad (5.10)$$

with scalar functions u_i . According to the theory of [6], the canonical operator (5.10) yields a scalar differential operator

$$\mathcal{L} = D^{2n} + \frac{1}{2} \sum_{i=1}^n ((u_i + r_i) D^{2n-2i} + D^{2n-2i} (u_i + r_i)), \quad (5.11)$$

where $r_i = r_i(u_1, \dots, u_{i-1})$ are differential polynomials in their arguments and particularly $r_1 = 0$. Hence the hierarchy (5.9) is equivalent the following system of Lax equations

$$\frac{\partial \mathcal{L}}{\partial t_k} = [(\mathcal{L}^{k/2n})_+, \mathcal{L}], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (5.12)$$

The Drinfeld-Sokolov hierarchy (5.9) of type $C_n^{(1)}$ carries a bi-Hamiltonian structure [6]. In [19] a set of Hamiltonian densities were selected appropriately such that they define a tau function, say, $\tilde{\tau}$ (to be distinguished from the notation τ of the CKP hierarchy above). With the same method as in [19] (see equation (5.13) there), we have

$$\partial_x^2 \log \tilde{\tau} = \frac{(-\Lambda \mid q^{\text{can}})}{(\Lambda \mid \Lambda^{-1})} = \frac{u_1}{2n}. \quad (5.13)$$

What is more, the Virasoro symmetries for the hierarchy (5.9) were constructed as (cf. [9])

$$\frac{\partial \tilde{\tau}}{\partial \beta_j} = V_j \tilde{\tau} + \tilde{\tau} O_j, \quad j = -1, 0, 1, 2, \dots \quad (5.14)$$

Here the operators

$$V_j = \frac{1}{4n} \sum_{k \in \mathbb{Z}^{\text{odd}}} : \tilde{p}_k \tilde{p}_{2n-j-k} : + \delta_{j0} c_n, \quad (5.15)$$

with c_n being a constant and

$$\tilde{p}_k = \frac{\partial}{\partial t_k}, \quad \tilde{p}_{-k} = k t_k \text{ for } k \in \mathbb{Z}_+^{\text{odd}}.$$

Choose

$$c_n = \frac{n}{24} \left(1 + \frac{1}{2n^2} \right), \quad (5.16)$$

then V_j satisfy the Virasoro commutation relation (see, for example, [11])

$$[V_i, V_j] = (i - j) V_{i+j}, \quad i, j \geq -1. \quad (5.17)$$

The terms O_j , determined by \mathcal{L}^{can} , are differential polynomials in second-order derivatives of $\log \tilde{\tau}$ with respect to the time variables. In particular, $O_{-1} = O_0 = 0$.

Generally, the terms O_j in (5.14) are called obstacles in linearizing Virasoro symmetries in [19]. The Virasoro symmetries are said to be linearized if all such O_j vanish, which is a crucial property of an integrable hierarchy of topological type [7]. We remark that all Drinfeld-Sokolov hierarchies associated to ADE-type affine Kac-Moody algebras, either untwisted or twisted, possess linearized Virasoro symmetries [19], see also [7, 18]. However, for the Drinfeld-Sokolov hierarchies of type C, it was unknown whether these obstacles O_j vanish or not, since it is not easy to compute them starting from the original definition in [19]. Such obstacles will be calculated alternatively in the forthcoming subsection in consideration that (5.12) is indeed a subhierarchy of the CKP hierarchy (2.4).

5.2 Non-linearizable Virasoro symmetries

Given an integer $n \geq 2$, unless otherwise stated the pseudo-differential operator (2.3) is henceforth assumed to satisfy

$$(L^{2n})_- = 0. \quad (5.18)$$

Under this constraint, the CKP hierarchy (2.4) is reduced to the hierarchy (5.12) with $\mathcal{L} = L^{2n}$, and the bilinear equation (2.11) becomes

$$\text{res}_z z^{2nj} w(\mathbf{t}; z) w(\mathbf{t}'; -z) = 0, \quad j \geq 0. \quad (5.19)$$

Meanwhile, the Poisson brackets (3.2) and (3.3) admit the constraint (5.18), hence one rederives the bi-Hamiltonian structure for the Drinfeld-Sokolov hierarchy achieved in [6]. The Hamiltonians are also given by the formulae (3.5), thus the tau function τ of the CKP hierarchy can be reduced to a tau function of the hierarchy (5.12).

Proposition 5.1 *For the Drinfeld-Sokolov hierarchy (5.12) of type $C_n^{(1)}$, the tau functions τ reduced from that of the CKP hierarchy and $\tilde{\tau}$ as recalled in the preceding subsection satisfy $\tau^2 = \tilde{\tau}$.*

Proof The proof is similar to that of Propositions 5.2 and 5.4 in [18] for Drinfeld-Sokolov hierarchies of types A and D. According to (3.6), (5.11) and (5.13), we have

$$\partial_x^2 \log \tau^2 = \text{res } L = \text{res } \mathcal{L}^{1/2n} = \frac{u_1}{2n} = \partial_x^2 \log \tilde{\tau}.$$

Hence

$$\partial_x^2 \left(\frac{\partial^2 \log \tau^2}{\partial t_k \partial t_l} - \frac{\partial^2 \log \tilde{\tau}}{\partial t_k \partial t_l} \right) = 0, \quad k, l \in \mathbb{Z}_+^{\text{odd}}.$$

Note that the terms in parentheses are differential polynomials in the coefficients of \mathcal{L} , namely, in (u_1, u_2, \dots, u_n) , hence their difference vanishes indeed. It follows that τ^2 and $\tilde{\tau}$ coincide. The proposition is proved. \square

Part of the additional symmetries (2.17) for the CKP hierarchy are compatible with the constraint (5.18). In fact, for $j \geq -1$, one has

$$\begin{aligned} \left(\frac{\partial L^{2n}}{\partial s_{1,2nj+1}} \right)_- &= [-(A_{1,2nj+1})_-, L^{2n}]_- \\ &= [-A_{1,2nj+1}, L^{2n}]_- \\ &= 4n(L^{2n(j+1)})_- = 0. \end{aligned} \quad (5.20)$$

Denote $s_j = 4n s_{1,2nj+1}$, then

$$\frac{\partial \mathcal{L}}{\partial s_j} = \frac{1}{4n} [-(A_{1,2nj+1})_-, \mathcal{L}] \quad (5.21)$$

are symmetries for the reduced hierarchy (5.12). Moreover, these symmetries satisfy the Virasoro commutation relation

$$\left[\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j} \right] = (j-i) \frac{\partial}{\partial s_{i+j}}, \quad i, j \geq -1 \quad (5.22)$$

when acting on \mathcal{L} or on the dressing operator Φ given as in (2.6). According to (4.14) we have

$$\frac{\partial \tau}{\partial s_j} = L_j \tau + \frac{\tau}{4n} \partial_x^{-1} T_{2nj+1}, \quad j \geq -1, \quad (5.23)$$

where

$$L_j = \frac{1}{8n} W_{2nj}^{(2)} + \delta_{j0} \frac{c_n}{2}. \quad (5.24)$$

Explicitly, one has

$$L_{-1} = \frac{1}{2n} \sum_{k \in \mathbb{Z}_+^{\text{odd}}} (k+2n) t_{k+2n} \frac{\partial}{\partial t_k} + \frac{1}{8n} \sum_{k+l=2n} k l t_k t_l, \quad (5.25)$$

$$L_0 = \frac{1}{2n} \sum_{k \in \mathbb{Z}_+^{\text{odd}}} k t_k \frac{\partial}{\partial t_k} + \frac{c_n}{2}, \quad (5.26)$$

$$L_j = \frac{1}{2n} \sum_{k+l=2nj} \frac{\partial^2}{\partial t_k \partial t_l} + \frac{1}{2n} \sum_{k \in \mathbb{Z}_+^{\text{odd}}} k t_k \frac{\partial}{\partial t_{k+2nj}}, \quad j \geq 1. \quad (5.27)$$

Here all indices k and l lie in $\mathbb{Z}_+^{\text{odd}}$, and the constant $c_n/2$ (see (5.16)) in L_0 is chosen for the validity of the following proposition.

Proposition 5.2 *For the Drinfeld-Sokolov hierarchy (5.12) of type $C_n^{(1)}$, the Virasoro symmetries (5.23) and (5.14) coincide. More precisely, acting on the tau function $\tilde{\tau}$ it holds that*

$$\frac{\partial \tilde{\tau}}{\partial s_j} = \frac{\partial \tilde{\tau}}{\partial \beta_j}, \quad j \geq -1. \quad (5.28)$$

Proof For $j \geq -1$, we write $L_j = L_j^{(2)} + L_j^{(1)} + L_j^{(0)}$, where $L_j^{(\nu)}$ is the part of the ν th order derivations in L_j . For instance,

$$L_0^{(0)} = \frac{c_n}{2}, \quad L_1^{(2)} = \frac{1}{2n} \sum_{k+l=2n} \frac{\partial^2}{\partial t_k \partial t_l}.$$

Similarly we write $V_j = V_j^{(2)} + V_j^{(1)} + V_j^{(0)}$ for V_j given in (5.15). It is easy to see

$$L_j^{(2)} = 2V_j^{(2)}, \quad L_j^{(1)} = V_j^{(1)}, \quad L_j^{(0)} = \frac{1}{2}V_j^{(0)}.$$

Since $\tilde{\tau} = \tau^2$, then

$$\begin{aligned} V_j \tilde{\tau} &= 4\tau V_j^{(2)} \tau + 2\tau V_j^{(1)} \tau + \tau V_j^{(0)} \tau - 2\tau^2 V_j^{(2)} \log \tau \\ &= 2\tau L_j \tau - \tilde{\tau} V_j^{(2)} \log \tilde{\tau}. \end{aligned}$$

Comparing (5.23) and (5.14), we have

$$\begin{aligned} &\partial_x \left(\frac{\partial \log \tilde{\tau}}{\partial s_j} - \frac{\partial \log \tilde{\tau}}{\partial \beta_j} \right) \\ &= \partial_x \left(\frac{2}{\tau} \left(L_j \tau + \frac{\tau}{4n} \partial_x^{-1} T_{2nj+1} \right) - \frac{1}{\tilde{\tau}} (V_j \tilde{\tau} + \tilde{\tau} O_j) \right) \\ &= \frac{1}{2n} T_{2nj+1} - \partial_x \left(O_j - V_j^{(2)} \log \tilde{\tau} \right). \end{aligned} \tag{5.29}$$

The left-hand side depends linearly on $\log \tilde{\tau}$, so does the right-hand side. Observe (4.23) and recall that O_j are differential polynomials in second-order derivatives of $\log \tilde{\tau}$, then the right-hand side of (5.29) must be of the form $\partial_x R_j \log \tilde{\tau}$ for some linear operator $R_j \in \mathbb{C}[\partial/\partial t_1, \partial/\partial t_3, \dots]$. Thus acting on $\log \tilde{\tau}$ one has

$$\frac{\partial}{\partial s_j} = \frac{\partial}{\partial \beta_j} + R_j, \quad j \geq 0,$$

where $R_{-1} = R_0 = 0$. In fact, all R_j must vanish by virtue of the Virasoro commutation relations for the symmetries $\partial/\partial s_j$ and for $\partial/\partial \beta_j$ respectively. Therefore

$$\frac{\partial \log \tilde{\tau}}{\partial s_j} = \frac{\partial \log \tilde{\tau}}{\partial \beta_j}, \quad j \geq -1. \tag{5.30}$$

The proposition is proved. \square

From the proof we also know that each T_{2nj+1} is a total derivatives of some differential polynomial in second-order derivatives of $\log \tilde{\tau}$ with respect to the time variables. Hence we obtain an alternative representation for the obstacles that were introduced from Kac-Moody-Virasoro algebra in [19].

Corollary 5.3 *The obstacles O_j in (5.14) can be represented as*

$$O_j = \frac{1}{2n} \left(\partial_x^{-1} T_{2nj+1} + \frac{1}{2} \sum_{k+l=2nj} \frac{\partial^2 \log \tilde{\tau}}{\partial t_k \partial t_l} \right), \quad j \geq 1, \tag{5.31}$$

where T_{2nj+1} are given in (4.15) with $\tau = \exp(\frac{1}{2} \log \tilde{\tau})$.

Theorem 5.4 *For the Drinfeld-Sokolov hierarchy of type $C_n^{(1)}$ with $n \geq 2$, the Virasoro symmetries (5.14) are non-linearizable. More precisely, in (5.14) $O_j \neq 0$ for any $j \geq 1$.*

Proof Substitute (4.23) into (5.31), then the part linear in $\log \tilde{\tau}$ of O_j with $j \geq 1$ is

$$\begin{aligned} O_j^{(1)} &= \frac{1}{4n} \left(-\frac{1}{2} \sigma_{2nj}(-\tilde{\partial}) + \sum_{k+l=2nj} \frac{\partial^2}{\partial t_k \partial t_l} \right) \log \tilde{\tau} \\ &= \frac{1}{4n} \sum_{k+l=2nj} \left(1 - \frac{1}{kl} \right) \frac{\partial^2 \log \tilde{\tau}}{\partial t_k \partial t_l} \\ &\quad - \frac{1}{8n} \sum_{\substack{k_1 m_1 + \dots + k_r m_r = 2nj \\ m_1 + \dots + m_r \geq 3; k_1 < \dots < k_r}} \left(\prod_{\nu=1}^r \frac{1}{m_\nu!} \left(\frac{2}{k_\nu} \frac{\partial}{\partial t_{k_\nu}} \right)^{m_\nu} \right) \log \tilde{\tau}. \end{aligned} \quad (5.32)$$

In particular, taking $j = 1$ one derives $O_1^{(1)} \neq 0$ hence shows $O_1 \neq 0$. Here it is used the fact that the flows $\partial/\partial t_k$ with $k = 1, 3, \dots, 2n-1$ in the hierarchy (5.12) are independent so that the linear part $O_1^{(1)}$ cannot be canceled by the omitted nonlinear part (cf. Remark 5.5 below).

Furthermore, provided $O_j = 0$ for some $j > 1$, it follows that $O_{j-1} = 0$ due to the commutation relation between $\partial/\partial s_{-1}$ and $\partial/\partial s_j$. Then step by step one deduces $O_1 = 0$, which is a contradiction. Therefore the theorem is proved. \square

Remark 5.5 The condition $n \geq 2$ in the above theorem is essential. Otherwise, suppose $n = 1$, then the reduced hierarchy (5.12) with

$$\mathcal{L} = D^2 + u$$

is nothing but the KdV hierarchy, or equivalently the Drinfeld-Sokolov hierarchy associated to the affine Kac-Moody algebra of type $A_1^{(1)}$. As it is known, the Virasoro symmetries for the KdV hierarchy is linearizable (see, for example, [17, 6, 19]).

In fact, according to (5.31) and (4.22), one has

$$O_1 = \frac{1}{2} \partial_x^{-1} T_3 + \frac{1}{4} \partial_x^2 \log \tilde{\tau} = -\frac{1}{4} \partial_x^2 \log \tilde{\tau} + \frac{1}{4} \partial_x^2 \log \tilde{\tau} = 0, \quad (5.33)$$

$$\begin{aligned} O_2 &= \frac{1}{2} \partial_x^{-1} T_5 + \frac{1}{2} \frac{\partial^2 \log \tilde{\tau}}{\partial x \partial t_3} \\ &= \frac{1}{3} \frac{\partial^2 \log \tilde{\tau}}{\partial x \partial t_3} - \frac{1}{12} \frac{\partial^4 \log \tilde{\tau}}{\partial x^4} - \frac{1}{2} \left(\frac{\partial^2 \log \tilde{\tau}}{\partial x^2} \right)^2. \end{aligned} \quad (5.34)$$

But the function $u = 2\partial_x^2 \log \tilde{\tau}$ satisfies the KdV equation

$$\frac{\partial u}{\partial t_3} = \frac{1}{4} \frac{\partial^3 u}{\partial x^3} + \frac{3}{2} u \frac{\partial u}{\partial x}.$$

One rewrites this equation in term of $\log \tilde{\tau}$ then achieves $O_2 = 0$. Furthermore, the Virasoro commutation relation for the symmetries $\partial/\partial s_j$ implies $O_j = 0$ for all $j \geq 3$. Thus the linearization of Virasoro symmetries for the KdV hierarchy is rederived, which coincides with the result in the literature. \square

Remark 5.6 The independence of the flows $\partial/\partial t_k$ with $k = 1, 3, \dots, 2n - 1$ implies that, the term T_{2n+1} for the reduced hierarchy (5.12) has the same expression with that for the CKP hierarchy (2.4). Thus T_{2n+1} in the latter case is also a total derivative of differential polynomial in second-order derivatives of $\log \tilde{\tau} = 2 \log \tau$ with respect to $t_1, t_3, \dots, t_{2n-1}$. Since n can be arbitrarily chosen, then such a property is possessed by every T_{2i+1} with $i \geq 1$ for the CKP hierarchy. In other words, T_{2i+1} is a total derivative of polynomial in at-least-second-order derivatives of $\log \tau$ with respect to the time variables. Therefore the claim at the end of Example 4.4 is proved. \square

6 Concluding remarks

We have defined a single tau function of the CKP hierarchy from its Hamiltonian densities. The tau function helps us to represent the bilinear equation of the hierarchy as well as its additional symmetries into some irregular form comparing to other cases studied in the literature. Especially, the actions on the tau function by additional symmetries involve strictly more than a central extension of the w_∞^C -algebra. For this tau function, its not clear yet whether there is a boson-fermion illustration as in [13]. Another interesting question is how to extend the present skills to the generalization [13] of the CKP hierarchy that contains both normal and super variables. An answer to this question must enrich our knowledge of integrable hierarchies and their applications.

By reducing additional symmetries for the CKP hierarchy, the Virasoro symmetries for the Drinfeld-Sokolov hierarchy associated to affine Kac-Moody algebra of type $C_n^{(1)}$ with $n \geq 2$ are rederived. The Virasoro symmetries coincide with those constructed in [19], and have been proved to be non-linearizable when acting on the tau function. In the proof we obtain a formula (5.31) to calculate the obstacles O_j . This formula, with its two sides arising from different contexts, still need to be better understood. We plan to study it in follow-up work.

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